COHERENT THEORIES AS DOUBLE LAWVERE THEORIES

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CAPE TOWN 2009

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GIVEN OPERATION SYMBOLS $\omega, \omega', \ldots$ WITH ARITIES

EQUATIONS BETWEEN TERMS $t = t'$

TERM:
- VARIABLE
- IF ω IS AN n-ARY OPERATION SYMB. AND $t_1, \ldots, t_n$ TERMS, THEN $\omega(t_1, \ldots, t_n)$ IS ALSO A TERM

EX: RINGS $+, \cdot, 0, 1, -$

$(x+y)\cdot(x-y) + 0$ IS A TERM

EQUATIONS

$x \cdot (y+z) = x \cdot y + x \cdot z$

ALGEBRA: SET A WITH OPERATIONS

$\omega : A^n \longrightarrow A$

A TERM $t(x_1, \ldots, x_m)$ IS INTERPRETED AS A FUNCTION

$|t| : A^m \longrightarrow A$

TO BE AN ALGEBRA A MUST SATISFY THE GIVEN EQUATIONS $t = t'$

$$\forall a_1, \dots, a_m \in A \left(|t|(a_1, \dots, a_m) = |t'|(a_1, \dots, a_m) \right)$$

HOMOMORPHISMS ARE FUNCTIONS PRESERVING THE OPERATIONS

LAWVERE THEORIES

CATEGORY $\mathbb{T}$: OBJECTS $[0], [1], [2], \dots$

- $[m] = [1] \times [1] \times \cdots \times [1]$
- MORPHISM $[m] \longrightarrow [1]$ IS EQUIVALENCE CLASS OF TERMS $t(x_1, \dots, x_m)$ WHERE TWO TERMS ARE EQUIVALENT IF THEIR EQUALITY FOLLOWS FORMALLY FROM THE GIVEN EQUATIONS.

AN ALG IS A PROD PRES FUNCTOR $\mathbb{T} \to \mathrm{SET}$.

A HOMOMORPHISM IS A NATURAL TRANSF.

FIRST ORDER LOGIC (FINITARY, ONE SORTED)

E.G. THE THEORY OF ORDERED FIELDS

- OPERATION SYMBOLS: $\omega, \omega', \dots$ WITH ARITIES ≥ 0

  E.G. $+, \cdot, 0, 1, -$

- PREDICATE SYMBOLS: $P, P', \dots$ WITH ARITIES ≥ 0

  E.G. $\leq$

- TERMS: - VARIABLES $x_0, x_1, x_2, \dots$

  - IF ω IS AN n-ARY OP SYMBOL AND $t_1, \dots t_n$ TERMS THEN $\omega(t_1, \dots t_n)$ TERM

  E.G. $x \cdot (y + x) + 0$

- FORMULAS: - ATOMIC:
  - $t_1 = t_2$
  - $P(t_1, \dots t_n)$

  E.G. $x(y+z) = xy + xz$

  $0 \leq x \cdot x$

- COMPOSITE: $\Phi \wedge \Psi$, $\Phi \vee \Psi$, $\Phi \to \Psi$, $\neg\Phi$, T, F, $\exists x\, \Phi$, $\forall x\, \Phi$

POSITIVE EXISTENTIAL: ONLY $\wedge, \vee, T, F, \exists$

COHERENT THEORY: A SET OF SEQUENTS

$\Phi \vdash \Psi$ Φ, Ψ P.E. FORMULAS

E.G. $T \vdash (xy)z = x(yz)$

$T \vdash x = 0 \vee \exists y\,(xy = 1)$

$0 \leq x \wedge 0 \leq y \vdash 0 \leq xy$

MODEL: - SET M

- FOR ω n-ARY OP. SYMBOL, $M(\omega): M^n \to M$
- FOR P n-ARY PRED. SYMBOL, $M(P) \subseteq M^n$

TERMS $t(x_1 \ldots x_n)$ INTERPRETED AS FNS $M(t): M^n \to M$

FORMULAS $\Phi(x_1 \ldots x_n)$ " " SUBSETS $M\Phi \subseteq M^n$

$\Phi \vdash \Psi$ HOLDS IN M IF $M\Phi \subseteq M\Psi$

M IS A <u>MODEL</u> IF ALL SEQUENTS IN THEORY HOLD IN M.

MORPHISM $f: M \to M'$ PRESERVES $M(\omega)$ & $M(P)$.

DOUBLE CATEGORIES

- OBJECTS, HORIZONTAL ARROWS, VERTICAL ARROWS, CELLS

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ v\downarrow & \alpha & \downarrow w \\ C & \xrightarrow[g]{} & D \end{array}$$

HORIZ COMPOSITION — CATEGORIES

VERT COMPOSITION — COHERENT ASSOCIATIVITY

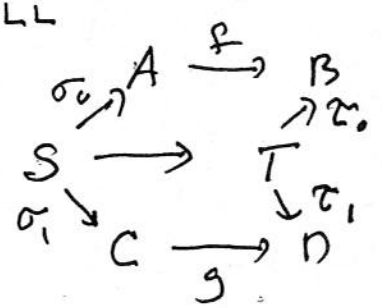
Ex: $\mathbb{R}$ING — RINGS, HOMOMORPHISM, BIMODULES

VERTICAL COMPOSITION = $\otimes$

Ex: $\mathbb{S}$ET$_s$ — SETS, FUNCTIONS, SPANS

SPAN $A \overset{S}{\longrightarrow} C$ is $A \xleftarrow{\sigma_0} S \xrightarrow{\sigma_1} C$

CELL



VERT COMP IS BY PULLBACK

Ex: $\mathbb{S}$ET$_r$ SETS, FUNCTION, RELATIONS

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ R\downarrow & ! & \downarrow S \\ C & \xrightarrow[g]{} & D \end{array}$$

IF $a \underset{R}{\sim} c \Rightarrow fa \underset{S}{\sim} gc$.

MAIN EXAMPLE:

LET $\mathcal{T}$ BE A FIRST ORDER COHERENT THEORY

DOUBLE CAT $\mathbb{T}$: OBJECTS $[0], [1], [2], \ldots$

- HORIZ. ARR. $[m] \longrightarrow [n]$ n-TUPLE $(t_1, \ldots, t_n)$ OF EQUIVALENCE CLASSES OF m-ARY TERMS. TERMS t, t' ARE EQUIVALENT IF $t = t'$ FOLLOWS FROM THE EQUATIONS IN $\mathcal{T}$.

- VERT. ARR. $[m] \longrightarrow\!\!\!\!\!\!\bullet\;\; [p]$ P.E. FORMULA $\Phi(x_1, \ldots, x_m; z_1, \ldots, z_p)$

- CELL

$$\begin{array}{ccc} [m] & \xrightarrow{(t_i)} & [n] \\ \Phi \big\downarrow & \vdash & \big\downarrow \Psi \\ [p] & \xrightarrow[(u_k)]{} & [q] \end{array}$$

UNIQUE ONE IF FROM $\mathcal{T}$ WE CAN PROVE $\Phi \vdash \Psi((t_i); (u_k))$

- VERT COMPOSITION $[m] \xrightarrow{\Phi} [p] \xrightarrow{\Theta} [n]$

$$\Theta \cdot \Phi (x_1, \ldots, x_m; z_1, \ldots z_n)$$

$$\equiv \exists y_1, \ldots, y_p \left(\Phi(x_1, \ldots, x_m; y_1, \ldots y_p) \wedge \Theta(y_1, \ldots, y_p; z_1, \ldots, z_n) \right)$$

DOUBLE FUNCTORS

$F : \mathbb{A} \longrightarrow \mathbb{X}$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle v}\downarrow & \alpha & \downarrow{\scriptstyle w} \\ C & \xrightarrow[g]{} & D \end{array} \quad \longmapsto \quad \begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle Fv}\downarrow & F\alpha & \downarrow{\scriptstyle Fw} \\ FC & \xrightarrow[Fg]{} & FD \end{array}$$

PRESERVES HORIZONTAL COMPOSITION

VERTICAL COMPOSITION

$$\begin{array}{c} A \\ {\scriptstyle v}\downarrow \\ C \\ {\scriptstyle x}\downarrow \\ E \end{array} \qquad \begin{array}{ccc} FA & = & FA \\ {\scriptstyle Fv}\downarrow & & \\ FC & \varphi_{x,v} & \downarrow{\scriptstyle F(x\cdot v)} \\ {\scriptstyle Fx}\downarrow & & \\ FE & = & FE \end{array}$$

φ SATISFIES COHERENCE CONDITIONS

F IS <u>STRONG</u> IF THE φ ARE HORIZONTAL ISOS

OTHERWISE IT IS <u>LAX</u>

EX: $\mathrm{INC} : \mathbb{S}\mathrm{ET}_r \longrightarrow \mathbb{S}\mathrm{ET}_s$ A LAX DOUB. FUNCT.

$\mathrm{IM} : \mathbb{S}\mathrm{ET}_s \longrightarrow \mathbb{S}\mathrm{ET}_r$ A STRONG DOUB. F.

FACT: IM IS LEFT ADJOINT TO INC IN THE APPROPRIATE DOUBLE CATEGORY SENSE.

MAIN EXAMPLE:

A MODEL OF $\mathcal{T}$ GIVES A STRONG FUNCTOR

$M : \mathbb{T} \longrightarrow \mathbb{S}\mathrm{ET}_r$.

$$M[n] = M^n$$

$$M[t_1, \dots t_m] = M^n \xrightarrow{\langle M(t_1), \dots M(t_m) \rangle} M^m \qquad \text{WELL-DEF}$$

$$M\Phi = M\Phi$$

THAT WE GET A STRONG FUNCTOR IS "SOUNDNESS".

A MORPHISM OF MODELS $f : M \longrightarrow M'$ GIVES A HORIZONTAL TRANSFORMATION OF STRONG FUNCTORS $\bar{f} : M \longrightarrow M'$ AND IN FACT ARE IN BIJECTION WITH THEM.

BINARY PRODUCTS

MODELS $M : \mathbb{T} \longrightarrow \mathbb{S}\mathrm{ET}_r$ SHOULD PRESERVE FINITE PRODUCTS

IN A DOUBLE CATEGORY

- $A_1 \times A_2$ HAS THE USUAL UNIVERSAL PROPERTY FOR HORIZONTAL ARROWS
- ALSO NEED $$\begin{array}{c} A_1 \times A_2 \\ {\scriptstyle v_1 \times v_2} \downarrow \\ B_1 \times B_2 \end{array}$$ WITH UNIVERSAL PROPERTY W.R.T CELLS
- REQUIRE $id_{A_1} \times id_{A_2} \cong id_{A_1 \times A_2}$
- $\times$ IS <u>STRONG</u> IF $(w_1 \cdot v_1) \times (w_2 \cdot v_2) \xrightarrow{\cong} (w_1 \times w_2) \cdot (v_1 \times v_2)$

EQUIV: $\mathbb{A} \xrightarrow{\Delta} \mathbb{A} \times \mathbb{A}$ HAS A STRONG RIGHT ADJ

OR: $\underline{A}_2 \rightrightarrows \underline{A}_1 \rightleftarrows \underline{A}_0$ IS IN <u>PROD</u>

<u>EX</u>: $\mathbb{S}\mathrm{ET}_r$ $\quad R_1 \times R_2 : A_1 \times A_2 \longrightarrow\!\!\!\bullet\; B_1 \times B_2$

FINITE PRODUCTS

GIVEN A DOUBLE CAT $\mathbb{A}$ CONSTRUCT $\mathbb{F}\mathrm{AM}_f^* \mathbb{A}$

- OBJECT: $(I, \langle A_i \rangle)$ I FIN SET, $A_i \in \mathbb{A}$

- HORIZ: $$\frac{(I, \langle A_i \rangle) \xrightarrow{(f, \langle h_j \rangle)} (J, \langle B_j \rangle)}{f : J \to I, \quad h_j : A_{f(j)} \to B_j}$$

- VERT:
$$\begin{array}{c} (I,A) \\ (S,v) \big\downarrow \\ (K,C) \end{array} \quad \Bigg| \quad \begin{array}{ccc} & & I \\ & \sigma_0 \nearrow & \\ S & & \\ & \sigma_1 \searrow & \\ & & K \end{array} \qquad \begin{array}{c} A_{\sigma_0 s} \\ v_s \big\downarrow \\ C_{\sigma_1 s} \end{array}$$

- CELL:
$$\begin{array}{ccc} (I,A) & \xrightarrow{(f,h)} & (J,B) \\ (S,v) \big\downarrow & (x,\alpha) & \big\downarrow (T,w) \\ (K,C) & \xrightarrow[f',h']{} & (L,D) \end{array} \quad \Bigg| \quad \begin{array}{ccc} I & \xleftarrow{f} & J \\ \nearrow & & \nearrow \\ S & \xleftarrow{x} & T \\ \searrow & & \searrow \\ K & \xleftarrow[f']{} & L \end{array} \qquad \begin{array}{ccc} A_{f\tau_0(t)} & \xrightarrow{h_{\tau_0 t}} & B_{\tau_0 t} \\ v_{xt} \big\downarrow & \alpha_t & \big\downarrow w_t \\ C_{f'\tau_1(t)} & \xrightarrow[h'_{\tau_1 t}]{} & D_{\tau_1 t} \end{array}$$

$\mathbb{A}$ HAS STRONG (LAX) FINITE PRODUCTS

IF $\Delta : \mathbb{A} \longrightarrow \mathbb{F}\mathrm{AM}_f^* \mathbb{A}$ HAS A STRONG (LAX)

RIGHT ADJOINT.

CONCRETELY

- FOR $A_1, \dots, A_n$ HAVE $\prod A_i \xrightarrow{p_i} A_i$ + UNIV. PROP.
- FOR $A_1, \dots A_n$ $\quad$ $\prod A_i \xrightarrow{p_{\sigma_0 s}} A_{\sigma_0 s}$ $\quad$ + UNIV. PROP.

$$
\begin{array}{ccccc}
A_1, \dots A_n & & \prod A_i & \xrightarrow{p_{\sigma_0 s}} & A_{\sigma_0 s} \\
v_1, \dots v_n \downarrow & & \prod v_s \downarrow & \pi_s & \downarrow v_s \\
C_1, \dots, C_m & & \prod C_k & \xrightarrow[q_{\sigma_1 s}]{} & C_{\sigma_1 s}
\end{array}
$$

<u>Ex</u>: PARALLEL PRODUCTS

$$
\begin{array}{ccccc}
A_1 & A_2 & & A_1 \times A_2 \\
v_1 \downarrow & v_2 \downarrow & \rightsquigarrow & \downarrow v_1 \times v_2 \\
C_1 & C_2 & & C_1 \times C_2
\end{array}
$$

SPAN IS $2 = 2 = 2$

<u>Ex</u>: LOCAL PRODUCTS

$$
\begin{array}{ccc}
A & & A \\
v_1 \downarrow \downarrow v_2 & \rightsquigarrow & \downarrow v_1 \times v_2 \\
C & & C
\end{array}
$$

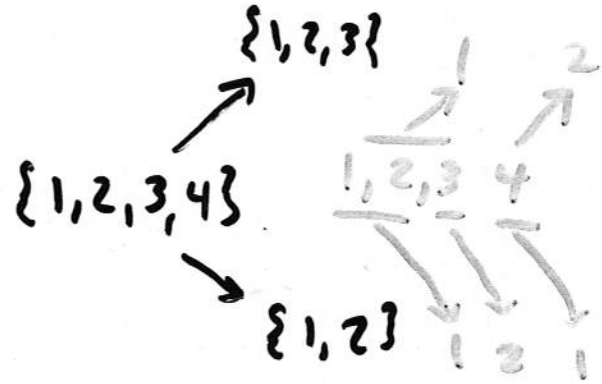
SPAN IS $1 \leftarrow 2 \rightarrow 1$

<u>Ex</u>: DIAGONALS

$$
\begin{array}{ccccc}
A & & & A \\
id_A \downarrow & \downarrow id_A & \rightsquigarrow & \downarrow \delta \\
A & A & & A \times A
\end{array}
$$

SPAN IS $1 \leftarrow 2 = 2$

EX: TAKE AS SPAN



4 PROJECTIONS

$$\begin{array}{ccc} A_1 \times A_2 \times A_3 & \xrightarrow{p_1} & A_1 \\ \prod_s v_s \downarrow & \pi_1 & \downarrow v_1 \\ C_1 \times C_2 & \xrightarrow[q_1]{} & C_1 \end{array}$$

$$\begin{array}{ccc} A_1 \times A_2 \times A_3 & \xrightarrow{p_1} & A_1 \\ \prod_s v_s \downarrow & \pi_2 & \downarrow v_2 \\ C_1 \times C_2 & \xrightarrow[q_1]{} & C_1 \end{array}$$

$$\begin{array}{ccc} A_1 \times A_2 \times A_3 & \xrightarrow{p_1} & A_1 \\ \prod_s v_s \downarrow & \pi_3 & \downarrow v_3 \\ C_1 \times C_2 & \xrightarrow[q_2]{} & C_2 \end{array}$$

$$\begin{array}{ccc} A_1 \times A_2 \times A_3 & \xrightarrow{p_2} & A_2 \\ \prod_s v_s \downarrow & \pi_4 & \downarrow v_4 \\ C_1 \times C_2 & \xrightarrow[q_1]{} & C_1 \end{array}$$

& UNIVERSAL

E.G. IN $\mathbb{S}\mathrm{ET}_r$ GIVEN $I \leftarrow S \rightarrow K$, SETS A_i, C_k

RELATIONS $R_s : A_{\sigma_0 s} \rightarrow C_{\sigma_1 s}$

$\prod R_s : \prod A_i \rightarrow \prod C_k$ IS THE RELATION

$$\langle a_i \rangle \underset{\prod R_s}{\sim} \langle c_k \rangle \iff a_{\sigma_0 s} \underset{R_s}{\sim} c_{\sigma_1 k} \text{ FOR ALL } s \in S$$

$$\iff \bigwedge_{s \in S} a_{\sigma_0 s} \underset{R_s}{\sim} c_{\sigma_1 k}$$

THIS IS <u>LAX</u> FUNCTORIAL.

<u>PROP</u>: $\mathbb{T}$ HAS LAX FINITE PRODUCTS, AND MODELS PRESERVE THEM. □

FINITE COPRODUCTS ARE DEFINED DUALLY

$\mathbb{F}\mathrm{AM}_f \mathbb{A}$

$$\frac{(I, A) \xrightarrow{(f,h)} (J, B)}{f : I \to J,\ h_i : A_i \to B_{fi}}$$

$$\mathbb{F}\mathrm{AM}_f^* \mathbb{A} = (\mathbb{F}\mathrm{AM}_f \mathbb{A}^{op})^{op}$$

LOCAL COPRODUCTS

$\mathbb{F}\mathrm{AM}_{f,l}\,\mathbb{A}$ FULL SUB DOUBLE CAT OF $\mathbb{F}\mathrm{AM}_f\,\mathbb{A}$ DETERMINED BY OBJECTS $(1, A)$.

$\mathbb{A}$ HAS STRONG (OPLAX) FINITE LOCAL COPRODUCTS IF $\Delta : \mathbb{A} \longrightarrow \mathbb{F}\mathrm{AM}_{f,l}\,\mathbb{A}$ HAS A STRONG (OPLAX) LEFT ADJOINT.

PROP: $\mathbb{T}$ HAS STRONG LOCAL COPRODUCTS (FINITE) AND MODELS PRESERVE THEM.

THEOREM: LET $\mathcal{T}$ BE A FIRST ORDER COHERENT THEORY AND $\mathbb{T}$ THE ASSOCIATED DOUBLE CATEGORY. THEN THE CATEGORY OF MODELS OF $\mathcal{T}$ IS EQUIVALENT TO THE CATEGORY OF STRONG FUNCTORS $M : \mathbb{T} \longrightarrow \mathbb{S}\mathrm{ET}_r$ THAT PRESERVE PRODUCTS AND LOCAL COPRODUCTS WITH HORIZONTAL TRANSFORMATIONS AS MORPHISMS.

SEMANTICS AS A FUNCTOR

AN IMPORTANT CONSEQUENCE OF FORMULATING A THEORY AS A CATEGORY IS THAT THERE IS A GOOD NOTION OF MORPHISM OF THEORIES.

LET $\mathbb{T}, \mathbb{T}'$ BE DOUBLE CATEGORIES WITH OBJECTS $[0], [1], \cdots$, FINITE PRODUCTS, FINITE LOCAL COPRODUCTS. A MORPHISM

$$F : \mathbb{T} \longrightarrow \mathbb{T}'$$

IS A STRONG DOUBLE FUNCTOR WHICH IS THE IDENTITY ON OBJECTS AND PRESERVES FINITE PRODUCTS AND LOCAL COPRODUCTS.

WE GET A CONTRAVARIANT FUNCTOR

$$\text{SEM} : \text{DOUBTH} \longrightarrow \text{CAT}/$$

$$\mathrm{SEM}(\mathbb{T}) = \begin{array}{c} \mathrm{MOD}(\mathbb{T}) \\ \big\downarrow U_{\mathbb{T}} \\ \mathrm{SET} \end{array}$$

NEGLECTING SIZE CONSIDERATIONS FOR NOW

WE GET A FUNCTOR IN REVERSE DIRECTION

$$\mathrm{STR} : \mathrm{CAT}/_{\mathrm{SET}} \longrightarrow \mathrm{DOUBTH}$$

$$\mathrm{STR}\left(A \xrightarrow{V} \mathrm{SET}\right) = \mathbb{T}_V$$

$\mathbb{T}_V$ OBJ : $[0], [1], \ldots$

HORIZ : $[m] \longrightarrow [n]$ ARE NAT. TRANSF. $V^m \to V^n$

VERT : $[m] \longrightarrow\!\!\!\!\!\bullet\;\; [n]$ ARE SUB FUNCTORS

$$R \rightarrowtail V^m \times V^n$$

CELLS : "INCLUSIONS"

$$\begin{array}{ccc} V^m & \xrightarrow{t} & V^n \\ R \big\downarrow\!\!\!\bullet & \subseteq & \big\downarrow\!\!\!\bullet\, S \\ V^p & \xrightarrow[u]{} & V^q \end{array}$$

$\mathbb{T}_V$ IS A FULL SUB DOUBLE CATEGORY

OF $\mathbb{R}\mathrm{EL}(\mathrm{SET}^A)$.

WE HAVE COMPARISONS

$$A \xrightarrow{\ \Phi\ } \mathrm{Mod}(\mathbb{T}_V)$$

$$V : A \longrightarrow \mathrm{Set}, \qquad U : \mathrm{Mod}(\mathbb{T}_V) \longrightarrow \mathrm{Set}$$

$$\Phi(A) : \mathbb{T}_V \longrightarrow \mathrm{Set}_r$$

$$\Phi(A)[n] = V(A)^n$$

$$\Phi(A)(t) = t(A) : V^m A \longrightarrow V^n A$$

$$\Phi(A)(R) = R(A) \subseteq V^m A \times V^n A$$

AND $\quad \Psi : \mathbb{T} \longrightarrow \mathbb{T}_U \quad (U : \mathrm{Mod}(\mathbb{T}) \longrightarrow \mathrm{Set})$

$$\Psi[n] = [n]$$

$$\Psi(h : [n] \to [m])(M) = M(h) : M^n \longrightarrow M^m$$

$$\Psi(r : [n] \rightsquigarrow [p])(M) = M(r) \subseteq M^n \times M^p$$

SMALLNESS

THERE MAY BE A PROPER CLASS OF NATURAL TRANSFs $t : V^m \to V^n$ OR OF $R \rightarrowtail V^m \times V^n$

$\mathrm{MOD}(\mathbb{T})$ IS ACCESSIBLE, HAS FILT. COLIMs

U PRESERVES FILTERED COLIMITS

LET $\mathcal{ACC}_f$ BE THE CATEGORY OF ACCESSIBLE CATEGORIES WITH FILTERED COLIMITS AND FUNCTORS PRESERVING THESE COLIMITS.

THEN $\quad \mathrm{SEM} : \mathrm{DOUBTH} \longrightarrow \mathcal{ACC}_f / \mathrm{SET}$

AND $\quad \mathrm{STR} : \mathcal{ACC}_f / \mathrm{SET} \longrightarrow \mathrm{DOUBTH}$

DEFINED AS BEFORE EXCEPT $R \rightarrowtail V^m \times V^n$ PRESERVES FILT COLIM.

THEOREM: STR AND SEM ARE ADJOINT ON THE RIGHT WITH UNITS GIVEN BY Φ AND Ψ. □

WHAT'S THE POINT?

- MAKES EXPLICIT THE ANALOGY WITH LAWVERE THYS
- PUTS LOGIC IN THE REALM OF DOUBLE CATEGORIES
  - POSSIBILITIES OF GENERALIZATION
    EG. MODELS IN $\mathbb{C}\mathrm{AT}_p$
  - $\Phi(x;z): [m] \to [n]$ COULD HAVE DIFFERENT CONDITIONS ON x AND z.
  - INFINITARY LOGIC $\mathbb{S}\mathrm{ET}_{\kappa,\lambda}$
- GIVES INSIGHT INTO DOUBLE CATEGORIES